

UNIFORM LIPSCHITZ REGULARITY OF FLAT SEGREGATED INTERFACES IN A SINGULARLY PERTURBED PROBLEM

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ABSTRACT. For the singularly perturbed system

$$\Delta u_{i,\beta} = \beta u_{i,\beta} \sum_{j \neq i} u_{j,\beta}^2, \quad 1 \leq i \leq N,$$

we prove that flat interfaces are uniformly Lipschitz. As a byproduct of the proof we also obtain the optimal lower bound near the flat interfaces,

$$\sum_i u_{i,\beta} \geq c\beta^{-1/4}.$$

1. MAIN RESULT

This note is intended as a remark on the recent paper of Soave and Zilio [6]. We study the flat segregated interfaces of the following singularly perturbed elliptic system

$$\Delta u_{i,\beta} = \beta u_{i,\beta} \sum_{j \neq i} u_{j,\beta}^2, \quad 1 \leq i \leq N. \quad (1.1)$$

Assume u_β is a sequence of positive solutions to this system in $B_1(0) \subset \mathbb{R}^n$, satisfying

$$\sup_{B_1(0)} \sum_i u_{i,\beta} \leq 1, \quad \forall \beta > 0.$$

By [5], $u_{i,\beta}$ are uniformly bounded in $\text{Lip}_{loc}(B_1(0))$. Hence we can assume $u_{i,\beta}$ converges to u_i in $C_{loc}(B_1(0))$. (It also converges strongly in $H_{loc}^1(B_1)$, see [7].) Then (u_i) satisfies the segregated condition

$$u_i u_j \equiv 0, \quad \forall \quad i \neq j.$$

It was proved in [7] (see also [4]) that the free boundary $\cup_i \partial\{u_i > 0\}$ has Hausdorff dimension $n - 1$ and it can be decomposed into two parts: $\text{Reg}(u_i)$ and $\text{Sing}(u_i)$. $\text{Sing}(u_i)$ is a relatively closed subset of $\cup_i \partial\{u_i > 0\}$ of Hausdorff dimension at most $n - 2$, while for any $x \in \text{Reg}(u_i)$, there exists a ball $B_r(x)$ such that there are only two components of (u_i) nonvanishing in this ball, say u_1 and u_2 . Furthermore, $u_1 - u_2$ is harmonic and $\nabla(u_1 - u_2) \neq 0$ in this ball. Hence the free boundary in this ball is exactly the nodal set of this harmonic function. In [6], it was proved that in this ball non-dominating species decay as follows:

$$\sum_{j \neq 1,2} u_{j,\beta} \leq C e^{-c\beta^c}. \quad (1.2)$$

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Without loss of generality and perhaps after taking a smaller r , we can assume $x = 0$ and $\{u_1 - u_2 = 0\} \cap B_r(0)$ is represented by the graph of a Lipschitz graph in the form $\{x_n = h(x')\}$, for $x' \in B_r^{n-1}(0)$.

Our main result is

Theorem 1.1. *The segregated interface $\{u_{1,\beta} = u_{2,\beta}\} \cap B_r(0)$ is represented by the graph of a Lipschitz function $x_n = h_\beta(x_1, \dots, x_{n-1})$, with the Lipschitz constant of h_β uniformly bounded. Moreover, h_β converges uniformly to h in $B_r^{n-1}(0)$.*

Some corollaries follow from the proof of this theorem.

Corollary 1.2. *There exists a constant $c_1 > 0$ independent of β such that*

$$|\nabla(u_{1,\beta} - u_{2,\beta})| \geq c_1, \quad \text{in } B_r(0). \quad (1.3)$$

Corollary 1.3. *There exists a constant $c_2 > 0$ independent of β such that,*

$$u_{1,\beta} + u_{2,\beta} \geq c_2 \beta^{-1/4}, \quad \text{in } B_r(0).$$

This improves the lower bound estimate in [6, Theorem 1.6] to the optimal one. Corollary 1.2 is also optimal, in the sense that there is no further uniform regularity of $\nabla u_{1,\beta} - \nabla u_{2,\beta}$. For example, $u_{1,\beta} - u_{2,\beta}$ does not converge to the limit in C^1 , see [6, Proposition 1.16].

The argument in this paper is similar to the proof for the regularity of flat interfaces in the Allen-Cahn equation presented in the second part of [10]. The main technical tool is the improvement of flatness estimate in [9]. In [9], this estimate is only stated for entire solutions. However, thanks to the local uniform Lipschitz estimate in [5], now we can show that it also holds for local solutions. Several new estimates from [6], especially the exponential decay of non-dominating species (1.2), also allows us to treat systems with more than two equations.

It is natural to conjecture that flat interfaces are also uniformly bounded in $C^{k,\alpha}$ for any $k \geq 1$ and $\alpha \in (0, 1)$. However, this is out of the reach of arguments in this note, which does not even imply any uniform $C^{1,\alpha}$ regularity. (In the Allen-Cahn equation, the uniform $C^{1,\alpha}$ regularity is only achieved by combining this argument with the result in [3].)

2. PROOF OF MAIN RESULTS

After restricting to a small ball, by a suitable translation and some scalings, we are in the following setting:

- (1) u_β is a sequence of solutions to (1.1) in $B_2(0)$;
- (2) u_β converges to $u := (u_1, u_2, 0, \dots, 0)$ uniformly in $B_2(0)$, and also strongly in $H_{loc}^1(B_2(0))$;
- (3) $u_{1,\beta}(0) = u_{2,\beta}(0)$;
- (4) there exists a small universal constant σ_0 (to be determined later) such that, for any $x \in B_1(0) \cap \{u_1 - u_2 = 0\}$,

$$\frac{\int_{B_1(x)} |\nabla u_1|^2 + |\nabla u_2|^2}{\int_{\partial B_1(x)} u_1^2 + u_2^2} \leq 1 + \sigma_0. \quad (2.4)$$

By multiplying u_β and u by a positive constant, we may assume

$$\int_{\partial B_1(0)} u_1^2 + u_2^2 = \int_{\partial B_1} x_n^2. \quad (2.5)$$

Because $u_1(0) - u_2(0) = 0$ and $u_1 - u_2$ is harmonic, by Almgren monotonicity formula for harmonic functions, we always have

$$\frac{\int_{B_1(x)} |\nabla u_1|^2 + |\nabla u_2|^2}{\int_{\partial B_1(x)} u_1^2 + u_2^2} \geq 1, \quad \forall x \in B_1(0) \cap \{u_1 - u_2 = 0\},$$

and (2.4) implies the existence of a unit vector e , which we assume to be the n -th coordinate direction, such that

$$\sup_{B_1(0)} (|u_1 - u_2 - x_n| + |\nabla(u_1 - u_2 - x_n)|) \leq c(\sigma_0) < 1/16, \quad (2.6)$$

provided σ_0 has been chosen small enough.

Some remarks are in order.

Remark 2.1. *In the following it is always assumed that (1.2) holds in $B_2(0)$. Then because $u_{i,\beta}$ is nonnegative and subharmonic, we get*

$$\sum_{i \neq 1,2} \int_{B_{3/2}(0)} |\nabla u_{i,\beta}|^2 \leq C e^{-c\beta^c}.$$

The following rescaling will be used many times in the proof:

$$u_{i,\beta}^\lambda(x) = \lambda^{-1} u_{i,\beta}(\lambda x), \quad \lambda > 0. \quad (2.7)$$

Once $\lambda > \beta^{-1/4}$, (1.2) still holds for $u_\beta^\lambda := (u_{i,\beta}^\lambda)$, perhaps with a larger C and a smaller c (but still independent of $\beta \rightarrow +\infty$).

Remark 2.2. *Throughout this section, we assume the Lipschitz constant of $u_{i,\beta}$ is bounded by a constant independent of β . Since all of the rescalings used in this paper are in the form (2.7), any rescaling of u_β has the same Lipschitz bound.*

Let us first recall some known results. The first one is the Almgren monotonicity formula, see for example [2, Proposition 5.2].

Proposition 2.3. *For any $x \in B_2(0)$,*

$$N(r; x, u_\beta) := \frac{r \int_{B_r(x)} \sum_i |\nabla u_{i,\beta}|^2 + \beta \sum_{i < j} u_{i,\beta}^2 u_{j,\beta}^2}{\int_{\partial B_r(x)} \sum_i u_{i,\beta}^2}$$

is increasing in $r \in (0, 2 - |x|)$.

By the strong convergence of u_β in $H_{loc}^1(B_2(0))$ and the bound (2.4), we can assume that, for all β large and $x \in \{u_{1,\beta} = u_{2,\beta}\} \cap B_1(0)$, $N(1; x, u_\beta) \leq 1 + 2\sigma_0$. Then by this proposition,

$$N(r; x, u_\beta) \leq 1 + 2\sigma_0, \quad \forall r \in (0, 1). \quad (2.8)$$

The next one is [8, Lemma 6.1] or [6, Theorem 1.1].

Lemma 2.4. *For any $x \in \{u_{1,\beta} = u_{2,\beta}\} \cap B_{3/2}(0)$,*

$$u_{1,\beta}(x) = u_{2,\beta}(x) \leq C\beta^{-1/4}.$$

The main technical result we will use is the following decay estimate, first proved in [9].

Theorem 2.5. *There exist four universal constants $\theta \in (0, 1/2)$, ε_0 small and K_0, C large such that, if u_β is a solution of (1.1) in $B_1(0)$, satisfying*

$$\sum_{i \neq 1, 2} \left[\sup_{B_1(0)} u_{i, \beta}^2 + \int_{B_1(0)} |\nabla u_{i, \beta}|^2 \right] \leq C e^{-c\beta^c}, \quad (2.9)$$

$$\varepsilon^2 := \int_{B_1(0)} |\nabla u_{1, \beta} - \nabla u_{2, \beta} - e|^2 \leq \varepsilon_0^2, \quad (2.10)$$

where e is a vector satisfying $|e| \geq 1/4$, and $\beta^{1/8} \varepsilon^2 \geq K_0$, then there exists another vector \tilde{e} , with

$$|\tilde{e} - e| \leq C(n)\varepsilon,$$

such that

$$\theta^{-n} \int_{B_\theta(0)} |\nabla u_{1, \beta} - \nabla u_{2, \beta} - \tilde{e}|^2 \leq \frac{1}{2} \varepsilon^2.$$

Proof. The proof is similar to [9, Theorem 2.2] with only three different points:

- (i) Now the system (1.1) could contain more than two equations. However, with the hypothesis (2.9) the effect of $u_{i, \beta}$ ($i \neq 1, 2$) is exponentially small, hence it does not affect the final conclusion.
- (ii) We do not claim [9, Lemma 3.4]. This estimate is used in [9, Eq. (5.1)]. Instead, we only provide a weaker estimate

$$\int_{B_{3/4}(0)} \beta u_{1, \beta} u_{2, \beta}^3 + \beta u_{2, \beta} u_{1, \beta}^3 \leq C \beta^{-1/8}. \quad (2.11)$$

This is the reason we replace the condition $\varepsilon^2 \gg \beta^{-1/4}$ in [9, Theorem 2.2] by a more restrictive one $\varepsilon^2 \gg \beta^{-1/8}$.

Note that $\beta u_{1, \beta} u_{2, \beta}^3 \leq u_{2, \beta} \Delta u_{1, \beta}$. Thus

$$\begin{aligned} \int_{B_{3/4}(0)} \beta u_{1, \beta} u_{2, \beta}^3 &\leq \int_0^{+\infty} \left(\int_{B_{3/4}(0) \cap \{u_{2, \beta} > t\}} \Delta u_{1, \beta} \right) dt \\ &\leq \int_0^{L\beta^{-1/4} \log \beta} \int_{B_{3/4}(0)} \Delta u_{1, \beta} + \int_{L\beta^{-1/4} \log \beta}^{+\infty} C e^{-c\beta^{1/4}t} \\ &\leq C \beta^{-1/4} \log \beta, \end{aligned}$$

where L is a large constant (fixed to be independent of $\beta > 0$) and we have used the fact that $\Delta u_{1, \beta} \leq C e^{-c\beta^{1/4}t}$ in $\{u_{2, \beta} > t\}$. (2.11) follows from this estimate if β is large enough.

- (iii) It is also not known whether [9, Lemma 3.3] holds. However, in [9] this estimate is only used to derive [9, Eq. (4.6)], which will be replaced by the following weaker estimate

$$\int_{B_{3/4}(0)} |\nabla u_{1, \beta}| |\nabla u_{2, \beta}| \leq C \beta^{-1/8}. \quad (2.12)$$

For simplicity, we will take a rescaling as in (2.7) so that $\beta = 1$ in the equation and the domain is $B_R(0)$ where $R = \beta^{1/4}$. Solutions are denoted by (u_i) .

Choose a T large so that $u_1 u_2 < T^2$ in $B_R(0)$ (see [8, Lemma 6.1]). By this choice $\{u_1 > T\}$ and $\{u_2 > T\}$ are disjoint.

For any $x \in \{u_1 < T, u_2 < T\}$, by the Lipschitz continuity of u_1 and u_2 , $u_1 \leq T + C$ and $u_2 \leq T + C$ in $B_1(x)$. Then by standard gradient estimates and Harnack inequality,

$$|\nabla u_i(x)| \leq C \sup_{B_1(x)} u_i \leq C u_i(x), \quad \forall i = 1, 2.$$

Thus by the Cauchy inequality,

$$\begin{aligned} \int_{B_{R-1}(0) \cap \{u_1 < T, u_2 < T\}} |\nabla u_1| |\nabla u_2| &\leq C \int_{B_{R-1}(0) \cap \{u_1 < T, u_2 < T\}} u_1 u_2 \\ &\leq C R^{\frac{n}{2}} \left(\int_{B_{R-1}(0)} u_1^2 u_2^2 \right)^{1/2} \\ &\leq C R^{n-1/2}, \end{aligned} \quad (2.13)$$

where we have used [8, Lemma 6.4], which implies

$$\int_{B_{R-1}(0)} u_1^2 u_2^2 \leq C R^{n-1}.$$

For $x \in \{u_1 \geq T\}$, by noting that

$$\Delta |\nabla u_2| \geq u_1^2 |\nabla u_2| - 2u_1 u_2 |\nabla u_1|,$$

we get

$$|\nabla u_2(x)| \leq C \sup_{B_{1/2}(x)} (u_1 u_2). \quad (2.14)$$

Because u_2 is subharmonic,

$$\sup_{B_{1/2}(x)} u_2 \leq C \int_{B_1(x)} u_2. \quad (2.15)$$

Since $u_1(x) \geq T$, by the Lipschitz bound on u_1 , if we have chosen T sufficiently large,

$$\frac{1}{2} \sup_{B_1(x)} u_1 \leq u_1(x) \leq \sup_{B_1(x)} u_1. \quad (2.16)$$

Combining (2.14)-(2.16) with the Lipschitz continuity of u_1 , we get

$$|\nabla u_1(x)| |\nabla u_2(x)| \leq C \int_{B_1(x)} u_1 u_2, \quad \forall x \in \{u_1 > T\} \cap B_{3R/4}(0).$$

Integrating this on $\{u_1 > T\} \cap B_{3R/4}(0)$ and using the Fubini theorem and the Cauchy inequality, we obtain

$$\begin{aligned} \int_{\{u_1 > T\} \cap B_{3R/4}(0)} |\nabla u_1| |\nabla u_2| &\leq C \int_{B_1(0)} \int_{\{u_1 > T\} \cap B_{3R/4}(0)} u_1(x+y) u_2(x+y) dx dy \\ &\leq C \int_{B_{\frac{3R}{4}+1}(0)} u_1 u_2 \\ &\leq C R^{n-1/2}. \end{aligned} \quad (2.17)$$

A similar estimate holds in $\{u_2 > T\} \cap B_{3R/4}(0)$. Combining (2.13) with these we get (2.12). \square

The next lemma can be used to show that the condition (2.10) is always satisfied for $(u_{i,\beta}^\lambda)$, provided $\lambda \gg \beta^{-1/4}$.

Lemma 2.6. *For any $\varepsilon > 0$, there exist two constants $K(\varepsilon)$ and $\delta(\varepsilon)$ so that the following holds. Suppose u_β is a solution of (1.1) in $B_2(0)$, with $\beta \geq K(\varepsilon)$, satisfying $u_{1,\beta}(0) = u_{2,\beta}(0)$, (2.9) and*

$$\frac{2 \int_{B_2(0)} \sum_i |\nabla u_{i,\beta}|^2 + \sum_{i < j} \beta u_{i,\beta}^2 u_{j,\beta}^2}{\int_{\partial B_2(0)} \sum_i u_{i,\beta}^2} \leq 1 + \delta(\varepsilon), \quad (2.18)$$

then there exists a vector e such that

$$\int_{B_1(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e|^2 \leq \varepsilon^2. \quad (2.19)$$

Proof. Assume by the contrary, there exists an $\varepsilon > 0$, a sequence of solutions u_β with $\beta \rightarrow +\infty$, satisfying $u_{1,\beta}(0) = u_{2,\beta}(0)$, (2.9) and

$$\limsup_{\beta \rightarrow +\infty} \frac{2 \int_{B_2(0)} \sum_i |\nabla u_{i,\beta}|^2 + \sum_{i < j} \beta u_{i,\beta}^2 u_{j,\beta}^2}{\int_{\partial B_2(0)} \sum_i u_{i,\beta}^2} \leq 1, \quad (2.20)$$

but for any vector e ,

$$\int_{B_1(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e|^2 \geq \varepsilon^2. \quad (2.21)$$

By our assumption, the Lipschitz constant of $u_{i,\beta}$ in $B_{3/2}(0)$ are uniformly bounded in β . By Lemma 2.4,

$$u_{1,\beta}(0) = u_{2,\beta}(0) \leq C\beta^{-1/4}.$$

Hence $u_{1,\beta}$ and $u_{2,\beta}$ are also uniformly bounded in $B_{3/2}(0)$. Assume it converges uniformly to $(u_1, u_2, 0, \dots)$ in $B_{3/2}(0)$. As before, $u_1 u_2 \equiv 0$ and $u_1 - u_2$ is a harmonic function. Moreover, $(u_{i,\beta})$ also converges to $(u_1, u_2, 0, \dots)$ in $H^1(B_1(0))$. Hence by Proposition 2.3 and (2.20), we obtain

$$\frac{\int_{B_1(0)} \sum_i |\nabla u_i|^2}{\int_{\partial B_1(0)} \sum_i u_i^2} \leq 1.$$

Then by the characterization of linear functions using Almgren monotonicity formula (noting that $u_1(0) - u_2(0) = 0$), we get a vector e such that

$$u_1(x) - u_2(x) \equiv e \cdot x, \quad \text{in } B_1(0).$$

By the strong convergence of $u_{i,\beta}$ in $H^1(B_1(0))$ again,

$$\lim_{\beta \rightarrow +\infty} \int_{B_1(0)} |\nabla u_{1,\beta} - \nabla u_1|^2 + |\nabla u_{2,\beta} - \nabla u_2|^2 = 0.$$

This is a contradiction with (2.21) and finishes the proof of this lemma. \square

After these preliminaries now we prove

Lemma 2.7. *For any $\sigma > 0$, there exist two universal constants $K_1(\sigma), K_2$ (K_2 independent of σ) such that the following holds. For any $x \in \{u_{1,\beta} = u_{2,\beta}\} \cap B_1(0)$, there exists an $r_\beta(x) \in (K_1\beta^{-1/4}, \theta)$ such that,*

- for any $r > r_\beta(x)$, there exists a vector $e(r, x)$, with $|e(r, x)| \geq 1/4$, such that

$$r^{-n} \int_{B_r(x)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e(r, x)|^2 \leq Cr^\alpha \int_{B_{3/2}(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e_n|^2, \quad (2.22)$$

where $\alpha = \log 2 / |\log \theta|$ and θ is as in Theorem 2.5;

- for $r \in (K_1\beta^{-1/4}, r_\beta(x))$, there exists a vector $e(r, x)$, with $|e(r, x)| \geq 1/4$, such that

$$r^{-n} \int_{B_r(x)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e(r, x)|^2 \leq K_2\beta^{-\frac{1}{8}}r^{-\frac{1}{2}}. \quad (2.23)$$

Moreover, for any $r \in (K_1\beta^{-1/4}, \theta)$,

$$|e(r, x) - e_n| \leq \sigma + C \left(\int_{B_{3/2}(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e_n|^2 \right)^{1/2} < 1/2, \quad (2.24)$$

for all β large.

Proof. Without loss of generality assume x is the origin 0. For each $k \geq 0$, let

$$E_k := \min_{e \in \mathbb{R}^n} \theta^{-kn} \int_{B_{\theta^k}(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e|^2,$$

which can be assumed to be attained by a vector e_k .

By our hypothesis, in particular (2.6), E_0 is very small for all β large. Moreover, e_0 is close to the n -th direction. In the following we will show that $|e_k| \geq 1/2$ up to scales $\theta^k \sim \beta^{-1/4}$.

Claim 1. For any $k \geq 0$, $E_k \geq \theta^n E_{k+1}$.

This is because, for any vector e ,

$$\theta^{-kn} \int_{B_{\theta^k}(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e|^2 \geq \theta^{-kn} \int_{B_{\theta^{k+1}}(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e|^2.$$

Let ε_0 be as in Theorem 2.5. Then choose σ_0 and \tilde{K}_1 according to Lemma 2.6 so that $2\sigma_0 \leq \delta(\varepsilon_0)$ and $\tilde{K}_1 \geq K(\varepsilon_0)$. By Lemma 2.6, we obtain

Claim 2. If $\beta^{1/4}\theta^k \geq \tilde{K}_1$, then $E_k \leq \varepsilon_0^2$.

In the following we take \tilde{k}_1 to be the largest k satisfying $\beta^{1/4}\theta^k \geq \tilde{K}_1$. k_1 is defined to be the largest $k \leq \tilde{k}_1$ so that for any $i \leq k$, $|e_i| \geq 1/2$.

Claim 3. For any $1 \leq k \leq k_1$, if $E_k \geq K_2\beta^{-1/8}\theta^{-k/2}$, where $K_2 = K_0\theta^{-n}$, then $E_k \leq \frac{1}{2}E_{k-1}$.

Let

$$\tilde{u}_{i,\beta}(x) := \theta^{1-k} u_{i,\beta}(\theta^{k-1}x),$$

which satisfies (1.1) with β replaced by $\beta_{k-1} := \beta\theta^{4k-4}$.

By Claim 2,

$$\varepsilon_{k-1}^2 := \int_{B_1(0)} |\nabla \tilde{u}_{1,\beta} - \nabla \tilde{u}_{2,\beta} - e_{k-1}|^2 = E_{k-1} \leq \varepsilon_0^2.$$

By Claim 1, $E_{k-1} \geq K_0\beta_{k-1}^{-1/8}$. Thus $\beta_{k-1}^{1/8}\varepsilon_{k-1}^2 \geq K_0$. Moreover, by definition we also have $|e_{k-1}| \geq 1/2$. Hence Theorem 2.5 applies, which implies the existence of a vector \tilde{e}_k such that

$$\theta^{-n} \int_{B_\theta(0)} |\nabla \tilde{u}_{1,\beta} - \nabla \tilde{u}_{2,\beta} - \tilde{e}_k|^2 \leq \frac{1}{2}\varepsilon_k^2.$$

Rescaling back, by the definition of E_k , we get **Claim 3**.

Note that in Claim 3, trivially we also have $E_{k-1} \geq E_k$. Thus we still have

$$E_{k-1} \geq K_2 \beta^{-\frac{1}{8}} \theta^{-\frac{k}{2}} \geq K_2 \beta^{-\frac{1}{8}} \theta^{\frac{1-k}{2}}.$$

Hence Claim 3 can be applied repeatedly. From this we deduce the existence of a k_2 such that, for any $k \geq k_2$, $E_k \leq K_2 \beta^{-1/8} \theta^{-k/2}$, while for any $k \leq k_2$, $E_k \geq K_2 \beta^{-1/8} \theta^{-k/2}$, and hence by Claim 3,

$$E_k \leq 2^{-1} E_{k-1} \leq \dots \leq 2^{-k} E_0.$$

It remains to show that $\theta^{k_1} \sim \beta^{-1/4}$. For $k \leq k_2$, because

$$\begin{aligned} \theta^{-kn} \int_{B_{\theta^k}(0)} |e_k - e_{k-1}|^2 &\leq 2\theta^{-kn} \int_{B_{\theta^k}(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e_k|^2 \\ &\quad + 2\theta^{-kn} \int_{B_{\theta^k}(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e_{k-1}|^2 \\ &\leq 2E_k + 2\theta^{-n} E_{k-1} \\ &\leq CE_0 2^{-k}, \end{aligned}$$

we get

$$|e_k - e_{k-1}| \leq CE_0^{\frac{1}{2}} 2^{-\frac{k}{2}}. \quad (2.25)$$

Similarly, for $k \geq k_2$,

$$|e_k - e_{k+1}| \leq C(n) K_2^{\frac{1}{2}} \beta^{-\frac{1}{16}} \theta^{-\frac{n+k}{4}}. \quad (2.26)$$

Let k_3 be the largest number satisfying

$$C(n) K_2^{\frac{1}{2}} \beta^{-\frac{1}{16}} \theta^{-\frac{n}{4}} \frac{\theta^{-\frac{k+1}{4}}}{\theta^{-1/4} - 1} \leq \sigma. \quad (2.27)$$

Note that by this choice, there exists a universal constant C such that

$$\frac{1}{C\sigma} \beta^{-\frac{1}{4}} \leq \theta^{k_3} \leq \frac{C}{\sigma} \beta^{-\frac{1}{4}}. \quad (2.28)$$

Adding (2.25) and (2.26) from $k = 0$ to k , we see for any $k \leq k_3$,

$$|e_k - e_0| \leq CE_0^{\frac{1}{2}} + \sigma < 1/4. \quad (2.29)$$

In particular, $|e_k| \geq 1/2$ for all $k \leq k_3$. Thus we can choose $k_1 \geq k_3$. By (2.28),

$$\theta^{k_1} \leq \frac{C}{\sigma} \beta^{-\frac{1}{4}}.$$

Finally, by choosing $K_1 := \max\{\tilde{K}_1, \theta^{k_3} \beta^{1/4}\}$ and $r_\beta := \theta^{k_2}$ we finish the proof. \square

Lemma 2.8. *For any $\varepsilon > 0$, there exists two constant $\tilde{\delta}(\varepsilon)$ and $\tilde{K}(\varepsilon)$ so that the following holds. Let u_β be a solution of (1.1) in $B_2(0)$ with $\beta \geq \tilde{K}(\varepsilon)$, satisfying $u_{1,\beta}(0) = u_{2,\beta}(0)$, (2.9) and*

$$\int_{B_2(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e|^2 \leq \tilde{\delta}(\varepsilon) \quad (2.30)$$

for some vector e with $|e| \geq 1/4$. Then $\{u_{1,\beta} = u_{2,\beta}\} \cap B_1(0)$ belongs to the ε neighborhood of $P_e \cap B_1(0)$, where P_e is the hyperplane orthogonal to e .

Proof. Assume by the contrary, there exists an $\varepsilon > 0$ and a sequence of solutions u_β in $B_2(0)$, with $\beta \rightarrow +\infty$, satisfying $u_{1,\beta}(0) = u_{2,\beta}(0)$, (2.9) and

$$\lim_{\beta \rightarrow +\infty} \int_{B_2(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e|^2 = 0, \quad (2.31)$$

where $|e| \geq 1/4$. (At first this vector may depend on β , but we can rotate $(u_{i,\beta})$ to make it the same one.) But there exists $x_\beta \in B_1(0) \cap \{u_{1,\beta} = u_{2,\beta}\}$ such that

$$\liminf_{\beta \rightarrow +\infty} \text{dist}(x_\beta, P_e) > 0. \quad (2.32)$$

Hence we can assume $x_\beta \rightarrow x_\infty$, which lies outside P_e .

By these assumptions and the uniform Lipschitz regularity of u_β , they are uniformly bounded in $\text{Lip}_{loc}(B_2(0))$ and can be assumed to converge to a limit (u_i) in $C_{loc}(B_2(0))$. By (2.9), $u_i \equiv 0$ for all $i \neq 1, 2$. By (2.31),

$$\int_{B_2(0)} |\nabla u_1 - \nabla u_2 - e|^2 = 0. \quad (2.33)$$

Hence by the main result in [7] and [4], $u_1 = (e \cdot x)^+$ and $u_2 = (e \cdot x)^-$.

Because $u_{i,\beta} \rightarrow u_i$ uniformly in $\overline{B_1}$, by the nondegeneracy of $u_1 - u_2$, we obtain a contradiction with (2.32). \square

Fix an $\varepsilon > 0$ and then choose a sufficiently small $\sigma \leq \tilde{\delta}(\varepsilon)/2$ and a sufficiently large $K_3 \geq \tilde{K}(\varepsilon)$ according to this lemma. By Lemma 2.7, Lemma 2.8 applies to u_β in $B_r(x)$ for $r \geq K_3\beta^{-1/4}$ (after scaling to the unit ball), which says $\{u_{1,\beta} = u_{2,\beta}\} \cap B_r(x)$ belongs to the εr neighborhood of $(x + P_{e(r,x)}) \cap B_r(x)$. Since $|e(r,x) - e_n| \leq 2\sigma$ (for β sufficiently large and e_n denotes the n -th direction), this implies $\{u_{1,\beta} = u_{2,\beta}\} \cap B_1(x) \subset \{|\Pi_{e_n}(y - x)| \leq C\sigma |\Pi_{e_n}^\perp(y - x)|\}$ once $|y - x| \geq K_3\beta^{1/4}$. Roughly speaking, this is equivalent to saying that $\{u_{1,\beta} = u_{2,\beta}\}$ is Lipschitz up to the scale $K_3\beta^{-1/4}$ in the direction e_n .

The next result shows that this Lipschitz property also holds for $r \in (0, K_3\beta^{-1/4})$.

Lemma 2.9. *For any $\delta > 0$ (sufficiently small) and $L > 0$, there exists an $R(\delta, L)$ so that the following holds. Suppose (u_i) is a solution of (1.1) with $\beta = 1$, in a ball $B_R(0)$ with $R \geq R(\delta, L)$, satisfying $u_1(0) = u_2(0)$,*

$$\sup_{B_L(0)} \sum_{i \neq 1, 2} u_i \leq Ce^{-cR^c}, \quad (2.34)$$

and

$$r^{-n} \int_{B_r(0)} |\nabla u_1 - \nabla u_2 - e|^2 \leq \delta, \quad \forall L < r < R, \quad (2.35)$$

where e is a unit vector. Then

$$\sup_{B_L(0)} |\nabla u_1 - \nabla u_2 - e| \leq c(n) < 1. \quad (2.36)$$

Moreover, $\{u_1 = u_2\} \cap B_L(0)$ is a Lipschitz graph in the direction e , with its Lipschitz constant bounded by $\bar{c}(\delta)$, which satisfies $\lim_{\delta \rightarrow 0} \bar{c}(\delta) = 0$.

Proof. Assume by the contrary, there exist δ and L , and a sequence of solutions $(u_{i,R})$ defined in $B_R(0)$ with $R \rightarrow +\infty$, satisfying (2.34) and (2.35), but the conclusion of this lemma does not hold.

Because $u_{1,R}(0) = u_{2,R}(0)$, by the Lipschitz bound, there exists a universal constant C such that

$$u_{1,R} = u_{2,R}(0) \leq C.$$

Combining this with (2.34) and the uniform Lipschitz bound on $u_{i,R}$, we see $(u_{i,R})$ are uniformly bounded in $\text{Lip}_{loc}(\mathbb{R}^n)$. Then using standard elliptic estimates and compactness results, we deduce that $(u_{i,R})$ converges to a limit (u_i) in $C_{loc}^2(\mathbb{R}^n)$, which is a solution of (1.1) with $\beta = 1$ in \mathbb{R}^n .

Passing to the limit in (2.34) gives $u_i(0) = 0$ for all $i \neq 1, 2$. Since $u_i \geq 0$, by the strong maximum principle, $u_i \equiv 0$ for all $i \neq 1, 2$. (2.35) can also be passed to the limit, which gives

$$r^{-n} \int_{B_r(0)} |\nabla u_1 - \nabla u_2 - e|^2 \leq \delta, \quad \forall r > L. \quad (2.37)$$

In particular, because e is nonzero, $(u_1, u_2) \neq 0$.

It is clear that (u_1, u_2) is a globally Lipschitz solution of the system

$$\Delta u_1 = u_1 u_2^2, \quad \Delta u_2 = u_2 u_1^2, \quad \text{in } \mathbb{R}^n. \quad (2.38)$$

Then the main result in [8] says $(u_1, u_2) = (g_1(\tilde{e} \cdot x), g_2(\tilde{e} \cdot x))$, where \tilde{e} is a vector and (g_1, g_2) is the one dimensional solution of (2.38). (It is essentially unique, see [1] and [2].) Substituting this into (2.37) we get

$$|\tilde{e} - e| \leq C\delta < 1/16,$$

provided δ has been chosen small enough. (Note that (2.38) has a scaling invariance, which however is fixed by the condition (2.37).)

By the implicit function theorem, for all R large, $\{u_{1,R} = u_{2,R}\} \cap B_L(0)$ is the graph of a smooth function h_R in the direction of \tilde{e} . By the convergence of $(u_{i,R})$ and the uniform lower bound on $\inf_{B_L(0)} |\nabla u_{1,R} - \nabla u_{2,R}|$, this function converges to 0 in a smooth way. The conclusion then follows. \square

Finally, we prove the two corollaries in Section 1.

Proof of Corollary 1.2. Take an arbitrary point x_0 . Let $\rho := \text{dist}(x_0, \{u_{1,\beta} = u_{2,\beta}\})$, which we assume to be attained at y_0 . Choose a k so that $\rho \in [\theta^{k+1}, \theta^k)$. (Notations as in the proof of Lemma 2.7.) Let

$$\tilde{u}_{i,\beta}(x) := \frac{1}{\rho} u_{i,\beta}(y_0 + \rho x).$$

If $\rho \leq K_3 \beta^{-1/4}$, (1.3) follows from (2.36) in Lemma 2.9.

If $\rho \geq K_3 \beta^{-1/4}$, (1.3) follows from (2.22) or (2.23) in Lemma 2.7 and standard interior elliptic estimates. (Note that in a neighborhood of $(x_0 - y_0)/\rho$ either $\tilde{u}_{1,\beta}$ or $\tilde{u}_{2,\beta}$ is very small compared to the other component.) \square

The proof of Corollary 1.3 is similar.

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